Numerical Methods 1. Fundamental concepts and relationships

How to consider a function to be „great” or „small”?

- In the function space $C[a,b]$ (the space of continuous functions defined on the interval $[a,b]$), introduce:

$$\| f \|_C := \max_{a \le x \le b} | f(x) | \quad \textbf{(maximum norm or C-norm)}$$

- Another possibility:

$$\| f \|_1 := \int_a^b | f(x) | \, dx \qquad (L_1\textbf{-norm})$$

- A third possibility: $\| f \|_2 := \sqrt{\int_a^b | f(x) |^2 \, dx} \qquad (L_2\textbf{-norm})$

… and there are a lot of other possibilities …

In all cases:

- $\| f \| \geq 0$, and $\| f \| = 0$ if and only if $f = 0$
- $\| \alpha \cdot f \| = | \alpha | \cdot \| f \|$
- $\| f + g \| \leq \| f \| + \| g \|$ („triangle inequality”)

How to consider two functions to be „near” or „far” from each other?

Define the **distance** of the functions f and g as: $\| f - g \|$

How to define a sequence of functions to converge to a function?

$f_n \to f$, if $\| f_n - f \| \to 0$.

How to consider a vector (i.e. an ordered *n*-tuple) to be „great” or „small”?

- In $\mathbf{R}^n$ (i.e. in the vector space of the ordered, real *n*-tuples), introduce:

$$\|\mathbf{x}\|_{\max} := \max_{1\le j\le n} |x_j| \quad \textbf{(maximum norm)}$$

- Another possibility:

$$\|\mathbf{x}\|_1 := \sum_{j=1}^{n} |x_j| \qquad \textbf{(1-norm} \text{ or } \textbf{sum norm)}$$

- A third possibility:

$$\|\mathbf{x}\|_2 := \sqrt{\sum_{j=1}^{n} |x_j|^2} \qquad \textbf{(2-norm} \text{ or } \textbf{Euclidean norm)}$$

… and there are a lot of other possibilities …

In all cases:

- $\| \mathbf{x} \| \geq 0$, and $\| \mathbf{x} \| = 0$ if and only if $\mathbf{x} = \mathbf{0}$
- $\| \alpha \cdot \mathbf{x} \| = | \alpha | \cdot \| \mathbf{x} \|$
- $\| \mathbf{x} + \mathbf{y} \| \leq \| \mathbf{x} \| + \| \mathbf{y} \|$ („triangle inequality”)

How to consider two vectors to be „near” or „far” from each other?

Define the **distance** of the vectors **x** and **y** as: $\| \mathbf{x} - \mathbf{y} \|$

How to define a sequence of vectors to converge to a vector?

$\mathbf{x}_n \to \mathbf{x}$, if $\| \mathbf{x}_n - \mathbf{x} \| \to 0$.

Vector spaces

A nonempty set X is a (real) **vector space**, if there exists and *addition* between the elements of X, and a multiplication between the elements of $\mathbf{R}$ and X such that the following *vector space axioms* are fulfilled:

For arbitrary $x, y, z \in X, \quad \lambda, \mu \in \mathbf{R}$:

- $x + y = y + x$ (the addition is commutative);
- $x + (y + z) = (x + y) + z$ (the addition is associative);
- there exists a zero vector $\mathbf{0}$ in X, such that $x + \mathbf{0} = x$ is valid for every $x \in X$;
- the addition is *invertible*, i.e. for every vector $x \in X$ there exists another vector $x_{-1} \in X$ such that their sum is the zero vector: $x + x_{-1} = \mathbf{0}$.
- $\lambda \cdot (\mu \cdot x) = (\lambda\mu) \cdot x$
- $\lambda \cdot (x + y) = \lambda \cdot x + \lambda \cdot y \qquad (\lambda + \mu) \cdot x = \lambda \cdot x + \mu \cdot x$
- $1 \cdot x = x$

The subset $X_0 \subset X$ is called **subspace**, if X_0 itself is also a vector space with respect to the operations defined in X.

Examples:

- The set of the real (complex) numbers **R** (**C**) is a vector space with respect to the ordinary real (complex) addition and multiplication.

- The set of ordered real pairs $\mathbf{R}^2$ is a vector space with respect to the componentwise operations:
  $(a,b)+(c,d):=(a+c,b+d), \qquad \lambda\cdot(a,b):=(\lambda a,\lambda b)$

- The set of ordered real triples $\mathbf{R}^3$ is a vector space with respect to the componentwise operations:
  $(a,b,c)+(u,v,w):=(a+u,b+v,c+w), \quad \lambda\cdot(a,b,c):=(\lambda a,\lambda b,\lambda c)$

- The set of ordered real n-tuples $\mathbf{R}^n$ is a vector space with respect to the componentwise operations:
  $(x_1,x_2,...,x_n)+(y_1,y_2,...,y_n):=(x_1+y_1,x_2+y_2,...,x_n+y_n)$
  $\lambda\cdot(x_1,x_2,...,x_n):=(\lambda x_1,\lambda x_2,...,\lambda x_n)$

- The set of $n\times m$ matrices $\mathbf{M}_{n\times m}$ is a vector space with respect to the componentwise operations:
  $[a_{kj}]+[b_{kj}]:=[a_{kj}+b_{kj}]$
  $\lambda\cdot[a_{kj}]:=[\lambda a_{kj}]$

Further examples:

- Let A be an arbitrary set. The set of all functions $f : A \to \mathbf{R}$ form a vector space with respect to the ordinary operations of functions:

  $(f + g)(x) := f(x) + g(x), \quad (\lambda \cdot f)(x) := \lambda \cdot f(x)$

- Let $[a, b]$ be a bounded, closed interval. The continuous functions defined on $[a, b]$ form a vector space (denoted by $C[a,b]$). In fact, this is a subspace of the vector space of all functions defined on $[a, b]$.

- Let $[a, b]$ be a bounded, closed interval. The functions defined on $[a, b]$ which are k times continuously differentiable form a vector space (denoted by $C^k[a,b]$). This is a subspace of $C[a,b]$. Moreover, $C^m[a,b]$ is a subspace of $C^k[a,b]$ provided that $m > k$.

Normed spaces

The vector space X is called **normed space**, if a function from X into **R** is defined (norm, denoted by $\|.\|$) such that the following norm axioms are fulfilled:

- $\| x \| \geq 0$
- $\| x \| = 0$ if and only if $x = \mathbf{0}$;
- $\| \alpha \cdot x \| = | \alpha | \cdot \| x \|$;
- $\| x + y \| \leq \| x \| + \| y \|$ (triangle inequality).

Examples:

- In **R**, define $\| x \| := | x |$; then the function $\|.\|$ fulfils the norm axioms.

- In **C**, define $\| z \| := | z |$; then the function $\|.\|$ fulfils the norm axioms. (Prove the triangle inequality!)

Examples in finite dimensional space:

In $\mathbf{R}^n$, define $\| x \|_{\max} := \max(| x_1 |,...,| x_n |)$; then the function $\| . \|_{\max}$ is a norm (**maximum norm**). The triangle inequality:

$$\| x+y \|_{\max} = \max_k | x_k + y_k | \le \max_k | x_k | + \max_k | y_k | = \| x \|_{\max} + \| y \|_{\max}$$

Another possibility: $\| x \|_1 := \sum_{k=1}^{n} | x_k |$ (**1-norm** or **sum norm**). The triangle inequality:

$$\| x+y \|_1 = \sum_{k=1}^{n} | x_k + y_k | \le \sum_{k=1}^{n} | x_k | + \sum_{k=1}^{n} | y_k | = \| x \|_1 + \| y \|_1$$

A third possibility: $\| x \|_2 := \sqrt{\sum_{k=1}^{n} | x_k |^2}$ (**2-norm** or **Euclidean norm**). The triangle inequality:

$$\| x+y \|_2^2 = \sum_{k=1}^{n} | x_k + y_k |^2 = \sum_{k=1}^{n} | x_k |^2 + 2\sum_{k=1}^{n} x_k y_k + \sum_{k=1}^{n} | y_k |^2 \le \| x \|_2^2 + 2 \| x \|_2 \cdot \| y \|_2 + \| y \|_2^2,$$

where we have utilized the *Cauchy inequality* (see later).

Examples in infinite dimensional spaces:

In $C[a,b]$, define $\| f \|_C := \max_{a \le x \le b} | f(x) |$; then the function $\|.\|_C$ is a norm (**maximum norm** or **C-norm**).

Another possibility: $\| f \|_1 := \int_a^b | f(x) | \, dx$, which defines a norm, too (L_1**-norm**).

A third possibility: $||f||_2 := \sqrt{\int_a^b |f(x|^2 \, dx}$ (L_2**-norm**). The triangle inequality:

$$\| f + g \|_2^2 = \int_a^b | f(x) + g(x) |^2 \, dx = \int_a^b | f(x) |^2 \, dx + 2\int_a^b f(x)g(x) \, dx + \int_a^b | g(x) |^2 \, dx \le$$

$$\le \| f \|_2^2 + 2\| f \|_2 \cdot \| g \|_2 + \| g \|_2^2 \quad ,$$

where we have utilized the *Cauchy-Schwarz-inequality*.

Banach spaces

Let X be a normed space. The vector sequence $(x_n) \subset X$ is said to be

- **bounded**, if the sequence $\| x_n \|$ is bounded, i.e. $\| x_n \| \leq C$ is valid for some $C \geq 0$;
- **converging** to the vector $x \in X$, if the sequence $\| x_n - x \|$ tends to 0, i.e. $\| x_n - x \| \to 0$;
- **Cauchy sequence**, if for every $\varepsilon > 0$ there is an index N such that $\| x_n - x_m \| < \varepsilon$ is valid for all indices $n, m \geq N$.

In arbitrary normed space X:
every convergent sequence is bounded,
and every convergent sequence is a Cauchy sequence.

The normed space X is **complete** or **Banach space**, if every Cauchy sequence is convergent in X.

Examples:

- Every finite dimensional normed space is a Banach space.
- The function space $C[a,b]$ is Banach space with respect to the C-norm, but not with respect to the L_1-norm.

Euclidean spaces, Hilbert spaces

The vector space X is said to be an **Euclidean space**, if a bivariate function is defined on X (**inner product** or scalar product, denoted by $\langle .,. \rangle$) such that the following properties are valid:

- $\langle x, x \rangle \geq 0$, and $\langle x, x \rangle = 0$ if and only if $x = \mathbf{0}$
- $\langle y, x \rangle = \overline{\langle x, y \rangle}$
- $\langle \alpha x, y \rangle = \alpha \cdot \langle x, y \rangle$
- $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$

Every Euclidean space X is a normed space with respect to the norm $\| x \| := \sqrt{\langle x, x \rangle}$ (the norm induced by the inner product).
Moreover, for every $x, y \in X$, the *Cauchy inequality* is valid:

$$| \langle x, y \rangle | \leq \| x \| \cdot \| y \|$$

The Euclidean space X is said to be a **Hilbert space**, if it is complete with respect to the norm induced by the inner product.

Every finite dimensional Euclidean space is also a Hilbert space.

Proof of the Cauchy inequality:
For simplicity, assume that the space is *real*. Then, for arbitrary vectors x, y:

$$\| x+y \|^2 = \| x \|^2 + 2\langle x, y\rangle + \| y \|^2$$

For arbitrary $x, y \in X$, $\alpha \in \mathbf{R}$, the inequality $\| x-\alpha y \|^2 \geq 0$ is valid, therefore:

$$\| x-\alpha y \|^2 = \| x \|^2 - 2\alpha\langle x, y\rangle + \alpha^2 \| y \|^2 \geq 0$$

Define $\alpha := \| x \| / \| y \|$, then we have: $\| x \|^2 - 2\dfrac{\| x \|}{\| y \|}\langle x, y\rangle + \dfrac{\| x \|^2}{\| y \|^2} \| y \|^2 \geq 0$,

whence $\langle x, y\rangle \leq \| x \| \cdot \| y \|$. Substituting $(-x)$ instead of x: $-\langle x, y\rangle \leq \| x \| \cdot \| y \|$ is also valid.

Proof of the triangle inequality (with respect to the norm induced by the inner product):

$$\| x+y \|^2 = \| x \|^2 + 2\langle x, y\rangle + \| y \|^2 \leq$$

$$\leq \| x \|^2 + 2\,|\langle x, y\rangle| + \| y \|^2 \leq$$

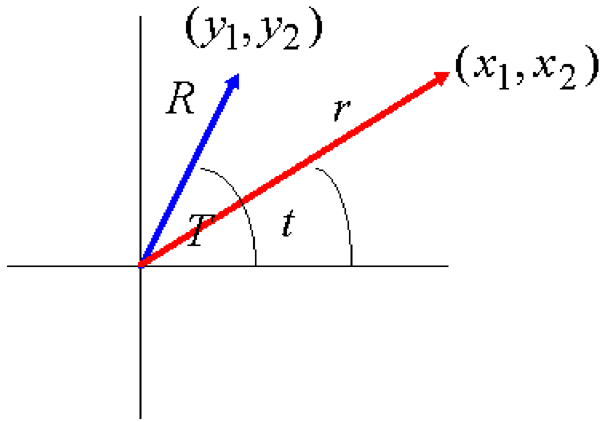
$$\leq \| x \|^2 + 2\| x \| \cdot \| y \| + \| y \|^2 = (\| x \| + \| y \|)^2$$

Examples:

- The space **R** is a Hilbert space with the inner product $\langle x, y\rangle := xy$.
- The space **C** is a Hilbert space with the inner product $\langle x, y\rangle := x\overline{y}$.
- The space $\mathbf{R}^2$ is a Hilbert space with the inner product $\langle x, y\rangle := x_1 y_1 + x_2 y_2$.

$$\langle (x_1, x_2), (y_1, y_2)\rangle = x_1 y_1 + x_2 y_2 = Rr\cos t\cos T + Rr\sin t\sin T = Rr\cos(T - t)$$

A geometrical illustration:



In short: $\langle x, y\rangle = \| x\| \cdot \| y\| \cdot \cos\theta$, where θ denotes the angle of the vectors x and y.

Euclidean spaces, examples:

The space $\mathbf{R}^n$ is a Hilbert space with the inner product $\langle x, y\rangle := \sum_{k=1}^{n} x_k y_k$ (which induces the Euclidean norm).

The form of the Cauchy inequality is: $\left|\sum_{k=1}^{n} x_k y_k\right| \le \sqrt{\sum_{k=1}^{n} |x_k|^2} \cdot \sqrt{\sum_{k=1}^{n} |y_k|^2}$.

Neither the maximum norm nor the sum norm can be induced by inner product.

The space $C[a,b]$ is an Euclidean space with the L_2-inner product $\langle f, g\rangle := \int_a^b f(x)\overline{g(x)}\,dx$, but it is not complete i.e. it is not a Hilbert space. However, the space $L_2(a,b)$ is a Hilbert space.

The form of the Cauchy inequality is: $\left|\int_a^b f(x)\overline{g(x)}\,dx\right| \le \sqrt{\int_a^b |f(x)|^2\,dx} \cdot \sqrt{\int_a^b |g(x)|^2\,dx}$

(Cauchy-Schwarz inequality)

Neither the maximum norm nor the L_1-norm can be induced by inner product.

Linear operators

Let X, Y be vector spaces. A mapping $A: X \to Y$ is said to be a **linear mapping** or **linear operator**, if it preserves the operations, i.e. $A(x+y) = A(x) + A(y)$ and $A(\lambda \cdot x) = \lambda \cdot A(x)$ are valid for every vectors $x, y \in X$ and scalar λ. If Y=**R** or **C**, then the mapping A is often called *linear functional.*

Examples:

- $A: \mathbf{R} \to \mathbf{R}$, $\quad Ax := a \cdot x$ (where $a \in \mathbf{R}$ is an arbitrary constant). Then A is a linear mapping.
- $D: C^1[a,b] \to C[a,b]$, $\quad Df := f'$. (the operator of the differentiation). Then D is a linear operator.
- Define $I: C[a,b] \to \mathbf{R}$, $\quad If := \int_a^b f(x)dx$. Then I is a linear functional.
- Let I be a finite, closed interval which contains the zero. Define $\delta: C[I] \to \mathbf{R}$, $\quad \delta f := f(0)$. Then δ is a linear functional. (*Dirac functional or* δ*-functional*)
- Let $A \in \mathbf{M}_{n \times n}$ be arbitrary. The mapping $\mathbf{R}^n \to \mathbf{R}^n$, $\quad x \to Ax$ is a linear mapping of $\mathbf{R}^n$ into itself (denoted by also A, if it causes no misunderstanding).

Boundedness and continuity

Let X, Y be normed spaces and let $A: X \to Y$ be a linear operator.

The linear operator $A: X \to Y$ is said to be **bounded**, if there exists a number $K \geq 0$ such that $\| Ax \| \leq K \cdot \| x \|$ is valid for every $x \in X$. The number K is a **bound** of the operator A.

A linear operator $A: X \to Y$ is bounded if and only if it is continuous everywhere.

The least upper bound of A is said to be the (operator) norm of A (denoted by $\| A \|$).

$$\| A \| = \sup\{\| Ax \| : x \in X , \| x \| \leq 1\}$$

In short: if the linear operator $A: X \to Y$ is continuous (i.e. bounded), then $\| Ax \| \leq \| A \| \cdot \| x \|$, and $\|A\|$ a least number with this property.

The operator norm depends on the norms of the spaces X and Y !

If $A: X \to Y$ and $B: Y \to Z$ are bounded then so is the product operator BA, and $\| BA \| \leq \| B \| \cdot \| A \|$

Examples for bounded (continuous) linear operators:

- The **identity operator** $I : X \to X$ is bounded, its norm is always 1 (independently of the norm of the space *X*).

- The **zero operator** is bounded, its norm is always 0 (independently of the norm of the space *X*).

If the spaces *X*, *Y* are *finite dimensional spaces*, then every linear operator $A: X \to Y$ is bounded.

- The operator of the differentiation $D: C^1[a,b] \to C[a,b], \quad Df := f'$ is bounded with respect to the norms of these spaces.

- The Dirac functional is bounded with respect to the $C(I)$-norm, but it is not bounded with respect to the L_1-norm.

Matrices and finite dimensional linear mappings

Let $A \in \mathbf{M}_{m\times n}$, and define the following mapping:

$$\mathbf{R}^n \to \mathbf{R}^m, \quad x \to Ax$$

Thus we have defined a $\mathbf{R}^n \to \mathbf{R}^m$ linear mapping (denoted by also A). For each matrix, there corresponds a linear operator and vice versa.

However, $\mathbf{M}_{m\times n}$ itself is a vector space. Therefore one can define several matrix norms on it.

Matrix norms which are NOT operator norms:

- Sum norm: $\| A \| := \sum_{k=1}^{n} \sum_{j=1}^{n} | a_{kj} |$
- Frobenius norm: $\| A \| := \sqrt{\sum_{k=1}^{n} \sum_{j=1}^{n} | a_{kj} |^2}$

These norms cannot be induced by vector norms (since the norm of the identity matrix differs from 1).

Matrix norms induced by vector norms (operator norms):

If both in $\mathbf{R}^m$ and in $\mathbf{R}^n$ the maximum norm is given, then the corresponding operator norm is the maximum of the row sums of the absolute values of the matrix entries (row norm):

$$\| A \|_{\max} = \max_{1 \le k \le m} \sum_{j=1}^{n} | a_{kj} |$$

$$\| Ax \|_{\max} = \max_{1 \le k \le m} \left| \sum_{j=1}^{n} a_{kj} x_j \right| \le \max_{1 \le k \le m} \sum_{j=1}^{n} | a_{kj} | \cdot | x_j | \le \left(\max_{1 \le k \le m} \sum_{j=1}^{n} | a_{kj} | \right) \cdot \left(\max_{1 \le j \le n} | x_j | \right) = \| A \|_{\max} \cdot \| x \|_{\max}$$

If both in $\mathbf{R}^m$ and in $\mathbf{R}^n$ the sum norm is given, then the corresponding operator norm is the maximum of the column sums of the absolute values of the matrix entries (column norm):

$$\| A \|_1 = \max_{1 \le j \le n} \sum_{k=1}^{m} | a_{kj} |$$

$$\| Ax \|_1 = \sum_{k=1}^{m} \left| \sum_{j=1}^{n} a_{kj} x_j \right| \le \sum_{k=1}^{m} \sum_{j=1}^{n} | a_{kj} | \cdot | x_j | = \sum_{j=1}^{n} \left(\sum_{k=1}^{m} | a_{kj} | \right) \cdot | x_j | \le \left(\max_{1 \le j \le n} \sum_{k=1}^{m} | a_{kj} | \right) \cdot \sum_{j=1}^{n} | x_j | = \| A \|_1 \cdot \| x \|_1$$

If both in $\mathbf{R}^m$ and in $\mathbf{R}^n$ the Euclidean norm is given, then the expression of the matrix norm is not simple. However, if A is *self-adjoint and positive definite*, then the operator norm equals to the maximal eigenvalue of the matrix.

Trigonometric Fourier series in $L_2(0,2\pi)$

An arbitrary real function $f \in L_2(0,2\pi)$ can be expressed as a trigonometric Fourier series which is convergent with respect to the $L_2(0,2\pi)$-norm:

$$f(x) = a_0 + \sum_{k=1}^{\infty} a_k \cos kx + \sum_{k=1}^{\infty} b_k \sin kx\,,$$

where the coefficients can be calculated as:

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} f(x)\,dx,$$

$$a_k = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos kx\,dx, \quad b_k = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin kx\,dx.$$

Complex exponential function

For any $z \in \mathbf{C}$, define $e^z := \sum_{k=0}^{\infty} \frac{z^k}{k!} = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots$ (**exponential series**)

The exponential series is absolutely convergent for each $z \in \mathbf{C}$. If z is real, then the sum equals to the value of the ordinary (real) exponential function.

(Euler's formula): For every $t \in \mathbf{R}$: $e^{it} = \cos t + i \sin t$.

Utilizing the well-known Taylor series of the sine and cosine functions:

$$\cos t = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \ldots, \qquad \sin t = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \ldots,$$

which implies that:

$$e^{it} = 1 + \frac{it}{1!} + \frac{i^2 t^2}{2!} + \frac{i^3 t^3}{3!} + \frac{i^4 t^4}{4!} + \frac{i^5 t^5}{5!} \ldots = 1 + \frac{it}{1!} - \frac{t^2}{2!} - \frac{it^3}{3!} + \frac{t^4}{4!} + \frac{it^5}{5!} \ldots$$

Separating the real and imaginary parts, we have the theorem.

The Discrete Fourier Transform (DFT)

If $f_0, f_1, f_2, \ldots, f_{N-1} \in \mathbf{C}$ is a finite sequence, then define its **discrete Fourier transform** as:

$$\hat{f}_k := \sum_{j=0}^{N-1} f_j e^{\frac{2kj\pi}{N} i} \qquad (k = 0,1,2,\ldots, N-1)$$

Relationship with the Fourier series: Let f be a continuous function defined on the interval $[0,2\pi]$, and denote by $f_j := f(\frac{2j\pi}{N})$. Then the sum $\frac{1}{N}\sum_{j=0}^{N-1} f_j e^{\frac{2kj\pi}{N} i}$ is a Riemann sum of the integral $\frac{1}{2\pi}\int_0^{2\pi} f(x)e^{ikx}dx$. Utilizing Euler's formula, we have:

$$\frac{1}{N}\hat{f}_k = \frac{1}{N}\sum_{j=0}^{N-1} f_j e^{\frac{2kj\pi}{N} i} \approx \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{ikx}dx = \frac{1}{2\pi}\int_0^{2\pi} f(x)\cos kx\,dx + i\cdot\frac{1}{2\pi}\int_0^{2\pi} f(x)\sin kx\,dx = a_k + ib_k$$

where a_k, b_k are the trigonometric Fourier coefficients.

Number of arithmetic operations: $O(N^2)$, which is too high!

The inverse Discrete Fourier Transform (iDFT)

Every finite sequence can be reconstructed from its DFT, namely:

$$f_k := \frac{1}{N}\sum_{j=0}^{N-1} \hat{f}_j e^{-\frac{2kj\pi}{N}i} \qquad (k = 0,1,2,\dots,N-1)$$

(inverse Discrete Fourier Transform)

For arbitrary $k = 0,1,2,\dots,N-1$, we have:

$$\frac{1}{N}\sum_{j=0}^{N-1} \hat{f}_j e^{-\frac{2kj\pi}{N}i} = \frac{1}{N}\sum_{j=0}^{N-1}\sum_{r=0}^{N-1} f_r e^{\frac{2rj\pi}{N}i} e^{-\frac{2kj\pi}{N}i} = \sum_{r=0}^{N-1} f_r \frac{1}{N}\sum_{j=0}^{N-1} e^{\frac{2(r-k)j\pi}{N}i} = \sum_{r=0}^{N-1} f_r \frac{1}{N}\sum_{j=0}^{N-1} z^j,$$

where $z := e^{\frac{2(r-k)\pi}{N}i}$. If $r = k$, then $z = 1$, therefore $\frac{1}{N}\sum_{j=0}^{N-1} z^j = 1$. If r differs from k, then z differs from 1, and the inner sum is the sum of a finite geometric sequence:

$$\sum_{j=0}^{N-1} z^j = \frac{z^N - 1}{z-1} = \frac{e^{2(r-k)\pi i} - 1}{z-1} = \frac{1-1}{z-1} = 0,$$ which completes the proof.

The Fast Fourier Transform (FFT)

Denote by $F_N : \mathbf{C}^N \to \mathbf{C}^N$ the (linear!) operator of the DFT:

$$(F_N f)_k := \sum_{j=0}^{N-1} f_j e^{\frac{2kj\pi}{N} i} \qquad (k = 0,1,\dots,N-1)$$

Assume that N is even: $N := 2N_1$. Let us separate the terms with even and odd indices in the expression of $(F_N f)_k$. First, let k be a 'small' index: $k = 0,1,\dots,N_1 - 1$:

$$(F_N f)_k = \sum_{l=0}^{N_1-1} f_{2l} e^{\frac{2\pi i}{N} k\cdot 2l} + \sum_{l=0}^{N_1-1} f_{2l+1} e^{\frac{2\pi i}{N} k\cdot(2l+1)} = \sum_{l=0}^{N_1-1} f_{2l} e^{\frac{2\pi i}{N} k\cdot 2l} + e^{\frac{2\pi i}{N} k} \sum_{l=0}^{N_1-1} f_{2l+1} e^{\frac{2\pi i}{N} k\cdot 2l} =$$

$$= \sum_{l=0}^{N_1-1} f_{2l} e^{\frac{2\pi i}{N_1} kl} + e^{\frac{2\pi i}{N} k} \sum_{l=0}^{N_1-1} f_{2l+1} e^{\frac{2\pi i}{N_1} kl}$$

The Fast Fourier Transform (FFT)

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Assume that N is even: $N := 2N_1$. Let us separate the terms with even and odd indices in the expression of $(F_N f)_k$. Now consider the 'big' indices having the form $N_1 + k$, where $k = 0,1,\ldots,N_1 - 1$:

$$(F_N f)_{N_1+k} = \sum_{l=0}^{N_1-1} f_{2l} e^{\frac{2\pi i}{N_1}(N_1+k)l} + e^{\frac{2\pi i}{N}(N_1+k)} \cdot \sum_{l=0}^{N_1-1} f_{2l+1} e^{\frac{2\pi i}{N_1}(N_1+k)l} =$$

$$= \sum_{l=0}^{N_1-1} f_{2l} e^{\frac{2\pi i}{N_1}kl} - e^{\frac{2\pi i}{N}k} \cdot \sum_{l=0}^{N_1-1} f_{2l+1} e^{\frac{2\pi i}{N_1}kl}$$

In both cases, both sums on the right-hand sides are discrete Fourier transforms with smaller vectors. The procedure can recursively be continued, if N is a power of two.

The recursive algorithm of the Fast Fourier Transform

- $N_1 := N/2, \quad f^{(0)} := (f_0, f_2, \ldots, f_{2N_1-2}), \qquad f^{(1)} := (f_1, f_3, \ldots, f_{2N_1-1})$

With recursive invocations, define

- $\hat{f}^{(0)} := F_{N_1} f^{(0)}, \qquad \hat{f}^{(1)} := F_{N_1} f^{(1)}$

- $$(F_N f)_k := \hat{f}_k^{(0)} + e^{\frac{2\pi i k}{N}} \cdot \hat{f}_k^{(1)} \qquad (k = 0,1,\ldots,N_1 - 1)$$
  $$(F_N f)_{N_1+k} := \hat{f}_k^{(0)} - e^{\frac{2\pi i k}{N}} \cdot \hat{f}_k^{(1)} \qquad (k = 0,1,\ldots,N_1 - 1)$$

where in the case of $N = 1$, we have $F_1 f := f$ (here f has one component only).

Number of arithmetic operations: $O(N \log N)$, which is much better than $O(N^2)$!

The algorithm can clearly generalized for the computation of the *inverse Discrete Fourier Transform* without difficulty.

Two-dimensional Discrete Fourier Transform

The DFT of a matrix $f \in \mathbf{M}_{N\times N}$ is the matrix $\hat{f} \in \mathbf{M}_{N\times N}$ with the following entries:

$$\hat{f}_{k,j} = \sum_{r=0}^{N-1}\sum_{s=0}^{N-1} f_{r,s} e^{\frac{2\pi ikr}{N}} e^{\frac{2\pi ijs}{N}}$$

Denote by F the 1D DFT, then:

$$\hat{f}_{k,j} = \sum_{r=0}^{N-1}\left(\sum_{s=0}^{N-1} f_{r,s} e^{\frac{2\pi ijs}{N}}\right) e^{\frac{2\pi ikr}{N}} = \sum_{r=0}^{N-1} (Ff_{r,.})_j e^{\frac{2\pi ikr}{N}}$$

The algorithm of the 2D DFT:

- For every row of the matrix *f*, substitute the 1D DFT of the corresponding row.
- For every column of this matrix, substitute the 1D DFT of the corresponding column.
- This results in the 2D DFT of the original matrix *f*.

The 1D DFT can be calculated by using the FFT algorithm. The total number of arithmetic operations is $O(N^2 \log N)$ (instead of the direct calculations, which are of $O(N^4)$!)